NASw-817

(NASA Contract NASw-817)

UNPUBLISHED PRELIMINARY DATA

29 p.

A Note on the Failure Modes of Filament Reinforced Materials, Including the Influence of Constituent Geometry and Properties

[1964] 29 refs

N64 13181

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CODE-1

(NASA CR-55192)

OTS: $2.60ph, $1.07mf

12

17 Conf

ABSTRACT

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Filament reinforced plate and shell structures are studied to define the relationship of constituent geometry and properties to the elastic constants of laminates, the buckling stress of filament wound cylindrical shells, and the lamina stresses in such shells. The results indicate the nature of desirable constituent property changes to achieve improved structural composites.

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OTS PRICE

XEROX $ 2.60ph

MICROFILM $ 1.07mf

For presentation at Meeting of Soc. of Plastics Ind., Chicago, 4-6 Feb. 1964

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INTRODUCTION

The development of very high strength and stiffness filaments, along with the development of the filament winding manufacturing techniques, offer the promise of obtaining efficient structural materials in the form of filament reinforced matrices for a variety of applications. An initial utilization of such materials is the current filament-wound, glass-reinforced plastic internal pressure vessel. In these vessels, the stiff high strength glass fibers are imbedded in a plastic matrix which performs the functions of fiber support and protection. The design of such vessels has been largely governed by isotensoid pressure vessel concepts wherein the structural role of the matrix material is neglected. In these structures, fibers are oriented so that all filaments are stressed uniformly and identically insofar as possible. For such an application, the desirable direction of materials development is clearly toward an improved strength to density ratio of the filament. However, new applications for filament reinforced materials will have stiffness requirements, either from deflection limitations or compressive stability considerations. For these applications, the matrix characteristics are of importance and in addition to the ultimate fiber strength, other failure modes must be considered. These include buckling and excessive shear stress in the matrix material. A study to determine the influence of fiber and matrix properties, as well as fiber orientation, upon the stiffness, internal stress distribution and buckling strength of filament reinforced materials is described in this paper. The results indicate desirable geometry for specified constituents and desirable constituent properties for specified applications. Hollow glass fibers as well as high modulus glass fibers and plastic binders are treated.

ANALYSIS

The aim of the analyses described herein is to determine the influence of constituent properties and geometry upon the failure of filament reinforced materials. The primary emphasis is placed upon the utilization of such materials for compressive applications inasmuch as this appears to be an area of potential application which has received relatively little attention. The first portion of this analysis section describes the method developed in ref. 1 and used herein to obtain the elastic constants of composites which contain fibers all oriented in a single direction. Next, these results are applied to filament wound laminates to evaluate the elastic constants of such materials. With the elastic properties defined as functions of the constituent properties and the laminate geometry, the elastic stability of a filament wound cylindrical shell under axial compression is treated. Finally, the average stresses in individual laminae are studied to evaluate potential shear failure modes. These various stages of the analysis are described individually in the following sections.

Elastic Constants for Uniaxially Stiffened Composites

The structural analysis of a non-homogeneous material such as filament wound glass reinforced plastic can be approached in several ways. Many studies have been based upon the so-called "netting" analysis which neglects the matrix material completely, and considers a net-like structure as he tensile load carrying structure. For compressive loads this technique is clearly inadequate and a more reasonable representation is to use an effective homogeneous but anisotropic material having properties which duplicate the average response of the actual inhomogeneous and isotropic material. This problem has been studied in ref. 1 where a fibrous composite consisting of a matrix reinforced by uniaxially oriented solid or hollow fibers was studied. Two cases were considered. In the first, the fibers are of identical cross section and form an hexagonal array in the transverse plane, and in the second, the fibers may have different diameters, but all have the same ratio of inner to outer diameter and are randomly located in the transverse plane. In both cases the composite is macroscopically homogeneous and transversely isotropic (these concepts will be discussed below) and has five elastic moduli. The problem then is to find expressions for the effective elastic moduli of the reinforced materials in terms of the elastic moduli and the geometric parameters of its constituents.

In fiber reinforced materials the ratio of length to fiber diameter is usually very large. Accordingly, fiber end conditions need only be considered in theSt. Venant sense. Consequently, it is sufficient to consider a very large cylindrical specimen of reinforced material, with fibers in the generator direction extending from base to base. If the average response of the material is considered, the composite properties can be represented by an effective homogeneous material having a Hooke's law defined by:

$$\bar{\sigma}_i = c_{ij}^* \bar{\epsilon}_j$$

where $\bar{\sigma}_i$ and $\bar{\epsilon}_j$ are the average values of the stress and strain components over the volume region, and the c_{ij}^* are the effective elastic constants. This definition of effective elastic moduli is phycially plausible; it is, however, not very useful because in order to find averages, a solution for the stress field has first to be found, which for this problem is a complex task. An equivalent and more fruitful approach is to define the effective elastic moduli in terms of strain energy and to bound the strain energy for simple applied average stress or strain fields, thus also bounding the elastic constant.

The general elastic features of the material are characterized by a principal elastic axis in the fiber direction and a transverse plane having isotropic properties. This stems from the fact that for the hexagonal fiber array, the reinforced material has hexagonal symmetry and is thus also transversely isotropic (see, e. g. ref. 2, p. 160) and for random fiber arrangement, a transverse isotropy is assumed. The stress-strain relation for transversely isotropic materials is thus the appropriate one and it may be written in terms of five elastic moduli. It is possible to select five independent moduli such that for specified states of stress and strain only one of these moduli will appear in the strain energy function. Thus, the bounds on strain energy from the minimum principles of the theory of elasticity can be used directly to yield bounds on the elastic moduli. Elastic constants computed by the bounding methods of ref. 1 are utilized herein in the evaluation of stresses and deflections of laminates of composite materials.

The upper and lower bounds for three of the required constants (longitudinal Young's modulus, E_1, and shear modulus, G_{12}, and Poisson's ratio in the lamina plane, ν_{21}) coincide. For the fourth constant, the transverse Young's modulus, E_2, the average of the upper and lower bounds has been used in the subsequent analyses. The fifth constant which may be taken as the transverse shear modulus, G_{23}, or Poisson's ratio in the transverse plane, ν_{23}, are not utilized in the laminate analysis which is essentially two dimensional.

Elastic Constants for Laminates

With the elastic constants of each plate defined, the properties of the laminate can be determined as a function of the individual plate characteristics.

This problem is treated in this section. This analysis plus the relationship between the elastic properties of the lamina or composite as a function of the properties and geometry of the constituents as obtained previously, enables the selection of desirable constituent characteristics. The analytical model will consider each layer or lamina as an orthotropic homogeneous material with defined elastic properties. It will be assumed that there are a large number of layers so that coupling between extensional and bending stresses may be neglected. Shear deformations in the plane of the plate will be treated, while those through the thickness will be neglected. First, the stress strain relations in one layer will be defined for arbitrary angles. Then the elastic constants in principle directions of the laminate will be evaluated. Finally, the constants of the laminate will be evaluated for arbitrary angles.

The problem of stresses in laminates has been studied extensively (see for example, ref. 3, 4, 5) and the methods used can be conveniently applied to obtain the desired elastic constants. A description of the analysis as used here is presented in appendix A, and the coordinate nomenclature for the laminate is shown in fig. 1. The analysis has been used to evaluate laminate constants primarily for four composite materials; namely, a reference E-glass reinforced epoxy (material I), a hollow E-glass fiber reinforced epoxy (material II), a high modulus glass reinforced epoxy (material III), and a high modulus binder reinforced with solid E-glass fibers (material IV). The properties of the constituents and the uniaxially reinforced composites for these four combinations are listed in Table I. A binder volume fraction of three-tenths has been used for all materials. Each of these four composites has been considered in laminate form with a two directional symmetric laminate with varying angle between layers, a two directional longitudinal and transverse laminate with varying proportions of thickness in each direction, and a triaxial laminate with equal 120^{o} angles between layers. The latter has the properties of an isotorpic material in the laminate plane.

The elastic constants for the two directional symmetric layup are shown in figs. 2 and 3 as a function of the semi-angle, θ , between the two directions. In fig. 2, it can be seen that for small values of θ , the longitudinal modulus, E_L is improved substantially by an increase in fiber modulus, E_f, and only slightly by an increase in binder modulus, E_b. For large values of θ , the reverse situation occurs. Note also that the longitudinal modulus, E_T, for laminates at 90^{o} minus the given angle. Thus equivalent conclusions can be drawn about transverse laminate moduli. The lowest curve presents the results for hollow fiber laminates having fibers with a ratio, α , of inner to outer radius of 0.8. Note that the density of this material is also substantially lower than that of all the other materials. Thus, to make a comparison of extensional stiffness on an equal weight basis the hollow fiber curve (material II) should be increased by 42.1%. Also shown are the longitudinal Young's moduli for each material for the isotropic laminate (0°, 60° and 120° layups). In fig. 3, the in-plane shear moduli, G_{LT}, are presented. All the shear moduli have maximum values when the fibers are oriented at 45° to the applied shear stresses. The values for the isotropic laminate are indicated. Also the values for 0°-90° layups are shown on the figure. These values are independent of the fraction in each direction and are equal to the value for uniaxial fiber orientation. Again differences between the materials considered are substantial and variable with geometry. It is apparent that an evaluation of the relative merits of the improvement of various properties requires consideration of a specific application. This is treated in the following section. The effect of fraction of material oriented in the load direction upon the longitudinal stiffness in a 0°-90°, or longitudinal-transverse, laminate is shown in fig. 4.

Buckling of Filament Wound Cylindrical Shells

Because fibrous composites are generally anisotropic materials, the evaluation of their efficiency for a given structural application is more comlex than for a isotropic material. That is, since one simple property of the material does not adquately define its performance, it is necessary to perform an analysis which includes the effect of all the material constants. Thus, the significance of the elastic constants defined in the previous section will be determined by studying a filament wound cylinder loaded in axial compression with elastic buckling as the failure criterion. A classical buckling analysis will be used (ref. 6) and is described in appendix B.

The principal results are plotted in fig. 5 in the form of buckling stress as a function of the helix angle with respect to the longitudinal axis. All shells considered are of equal weight so that the shells of material II are of greater thickness than all others. The results for material II have been multiplied by the ratio of material II thickness to material I thickness so that the comparison is on a load carrying basis for equal weight. The shells have all helical windings. The results are symmetric with respect to 45° and show maximum loads for fibers at about 20° or 70° to the longitudinal axis. For the reference E-glass reinforced plastic, material I, the variation with respect to the lamina angle is relatively small. Other fiber geometries have also been treated. For circumferential and longitudinal windings the highest buckling load was achieved for one half the fibers in each direction. This load was equal to the buckling load for helical windings at $\theta = \pm 45^{\circ}$. This result was observed for four materials considered. For material I, the range of values of buckling stress for the 0°-90° windings ranged from 11.4 to 12.3 ksi. The triaxial or isotropic winding was also considered. For all materials this was found to ve the highest buckling stress, as shown in Table II.

TABLE II

Isotropic Buckling Loads

Material	cr
I	16.5 ksi
II	15.0*
III	22.3
IV	20.6

*Adjusted for density difference

Note that for material I, the improvement of the buckling load for isotropic windings over the highest buckling load for helical windings is larger than the variation obtained over the range of all other goemetries considered. This result was also obtained for all other materials treated. This effect of geometry is in some ways disappointing. That is, given the opportunity to orient material to achieve maximum buckling resistance to a uniaxial load, the best choice appears to be a random or isotropic array. Secondly, all other geometric arrays seem to have a small effect on structural behavior relative to each other. One note of caution is in order; namely, that a small deflection, classical analysis has been used as the basis of this study. However, the experiments of ref. 7 indicated reasonable agreement with a classical analysis for shells of these dimensions. The shells are of relatively small radius to thickness ratio and filament wound shells are relatively free of initial imperfections in the shell shape so that there is justification for using this type of analysis.

As for the variation in fiber geometry, the hollow fibers are seen to present no improvement over the solid fibers for this application. This is in marked contrast with the results obtained for column buckling and other uniaxial orientation applications. (See ref. 8.) A more significant effect is observed for the use of high modulus glass fibers. Also it is seen that the highest results for helical laminates are achieved with the use of a high modulus binder material. This large increase in buckling strength is achieved by doubling the modulus of the reference epoxy binder, an approach which appears to warrant experimental investigation. Doubling the modulus of an epoxy by the addition of micron size particles of alumina or calcium carbonate is relatively simple and has been done experimentally with about 30% by volume, particle material. The effect of variation of the constituent moduli upon the buckling of a helically wound shell of 15° wrap angle is shown in fig. 6. The benefits of varying fiber modulus are greater than the benefits for the same percentage increase of binder modulus, but the quantity is much more susceptible to control.

Note that all the computed buckling stresses are low relative to the compressive strength of the material.

For comparison, the effective buckling stress for an equal weight magnesium shell corrected for the density of difference, would be 41.5 ksi, which is a much higher fraction of the material strength. It appears that proper utilization of glass reinforced plastics for compression buckling applications requires a change in geometry, to a sandwich construction for example, in order to utilize the high compressive strength of the material and compete with a metallic shell.

Lamina Stresses

The consideration of failure modes for fibrous composites requires consideration of the transverse and shear stresses in addition to the stresses in the fiber direction. This is of relatively greater importance for compressive applications, than for tensile applications because the "netting" action which can resist tensile loads will not be adequate to resist compressive loads without the relatively rigid support of the binder material. For simple loading conditions, like uniaxial tension of a symmetric laminate, interlaminar stresses around the edges of the laminate transmit shear stresses between the layers. These stresses are edge stresses and are not readily susceptible to analysis. They do introduce shear stresses into each lamina which must be considered as a potential cause of failure. The direction parallel to the fibers appears to have the lowest resistance to shear stresses and thus attention will be concentrated on shear stresses in that direction. The stresses in the lamina plane transverse to the fiber direction can produce shear stresses through the lamina thickness which are a potential cause of shear failure. The three components of the lamina stresses in the principal directions of each lamina are thus the stresses of interest. These are evaluated by the procedure of appendix A.

The shear stress in the laminate plane parallel and normal to the fibers is shown in fig. 7 for materials I and II as a function of the lamina angle. The stress is normalized with respect to the average applied extensional stress. It is seen that these stresses vary considerably with geometry and that they can represent a large fraction of the applied stress. The effect of constituent properties is more clearly indicated in fig. 8 where the influence of the binder modulus is studied for a 15° laminate. It is seen that the increase in binder modulus which is desired to improve buckling resistance results in an increased lamina shear stress. However, this variation is much less than the variation produced by changing the lamina angle. The mode of failure associated with these high shear stresses requires some experiment study. However, an indication of what may be encountered is indicated by the computed results.

indicating the importance of the various failure modes as shown in fig. 9. Here it has been assumed that a lamina shear stress or a transverse shear stress of 5 ksi shall be considered failure. The allowable axial stress for a cylindrical shell of radius to thickness ratio of 145 is shown in the figure. It is seen that each of the failure modes occurs in a given geometry range. A change in material properties to those of material III changes the buckling stress, but has a minor influence on the stress level or region of importance for the shear modes. Failures due to transverse shear have been experimentally observed. (See ref. 7.) Failures due to lamina and interlaminar shear may interact considerably. The effect of geometry appears to be more substantial than that due to elastic material properties. In the area of shear stress failures it appears that more precise analytical tools must be used to define failure criteria, but it is clear from these approximate studies that materials of improved shear strength are essential for high performance compressive applications of reinforced plastics.

CONCLUSIONS

The application of glass reinforced plastic laminates to compressive structural applications has been studied. The influence of fiber and matrix elastic moduli and geometry has been evaluated in three major areas. First the elastic constants of laminates have been evaluated with the major conclusion being that wide variations in any one elastic constant can easily be acheived and that evaluation of the significance of these properties requires the definition of a specific application. The second phase of the study treated a filament wound cylindrical shell in axial compression as the typical application to provide the evaluation of the various material properties encountered in the first phase. The use of relationships between the composite properties and the constituent properties made possible the treatment of the influence of changes in fiber or binder modulus upon the structural performance of the composite laminate. The elastic stability studies showed that wide changes in filament orientation had a small effect on composite performance and that the isotropic laminate was clearly the most efficient for the application considered. Hollow and solid E-glass fibers were shown to be essentially similar in performance, while improvement of either the fiber or binder modulus was shown to have a significant and favorable effect upon the buckling stress. All buckling stresses were relatively low when compared to the compressive strength of these materials and it appears that a change in geometry such as the use of sandwich construction is a beneficial area of study. The average stresses in individual laminae were also studied. Primary consideration was given to shear induced failure modes. The in-plane shear stress was found to be more sensitive to fiber orientation than to matrix or fiber properties. Variation of constituent properties and geometry can vary the failure mode to either shell buckling, transverse shear failure or in-plane shear failure. The interactions of these modes requires experimental study, but the stress magnitudes indicate the need for materials of improved shear strength.

It appears that rational methods exist, not only for evaluating structural performance of a given material, but also for the significant purpose of indicating the nature of the change in constituent properties desirable to produce improved structural composites.

APPENDIX A
Laminate Stress Analysis

The analysis of stresses in a laminate follows that of ref. 4 but is modified to evaluate elastic constans and simplified to neglect coupling between bending and extension. The laminate is considered to have a large number of symmetric laminae so that the bending stiffness and extensional stiffness are related in the same fashion as they are for an homogeneous material. This also results in bending and extensional stresses being uncoupled. Transverse shear is also neglected.

The stress-strain law for each layer relative to the lamina principal axes is:

$$\sigma_i = c_{ij} \epsilon_j \qquad i, j = 1, 2, 3 \qquad \text{(A1)}$$

σ_i = stress
ϵ_j = strain
c_{ij} = elastic constants
- and a repeated index denotes summation

For the orthotropic lamina of a filament wound material:

$$c_{ij} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \qquad \text{(A2)}$$

where these stiffnesses are related to the conventional elastic constants by

$$c_{11}^{(k)} = \frac{E_1^{(k)}}{1 - \nu_{21}^{(k)} \nu_{12}^{(k)}}$$

$$c_{22}^{(k)} = \frac{E_2^{(k)}}{1 - \nu_{21}^{(k)} \nu_{12}^{(k)}} \qquad \text{(A3)}$$

where the L_{ij} are the elements of the inverse of the T_{ij} matrix.

Consider a laminate of n layers subject to in-plane loads. Since transverse shear and coupling between bending and extension have been neglected the strains in all layers will be the same and the average stresses, $\bar{\tau}_i$, will be:

$$\bar{\tau}_i = \sum_{k=1}^{n} \bar{\sigma}_i^{(k)} t_k$$

$$\bar{\tau}_i = \sum_{k=1}^{n} c_{ij}^{(k)} \bar{\epsilon}_j t_k \tag{A7}$$

$$\bar{\tau}_i = \bar{\epsilon}_j \bar{A}_{ij}$$

where t_k = fraction of total thickness in kth layer and

$$\bar{A}_{ij} = \sum_{k=1}^{n} \bar{c}_{ij}^{(k)} t_k$$

Equations (A7) may be rewritten

$$\bar{\epsilon}_j = \bar{B}_{ij} \bar{\tau}_i \tag{A8}$$

where the B_{ij} are the elements of the inverse of the A_{ij} matrix.

This is the solution for lamina strains as a function of applied stresses, .

From Equation (A8) the desired elastic constants can be defined as follows:

$$E_L = \frac{1}{\bar{\bar{B}}_{11}}$$

$$c_{12}^{(k)} = c_{21}^{(k)} = \frac{\nu_{21}^{(k)} E_2^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}$$

$$c_{33}^{(k)} = 2G_{12}$$

E_1 = Young's modulus in fiber direction
E_2 = Youn'gs moudlus normal to fiber direction
G_{12} = Shear modulus in fiber plane
ν_{12} = Ratio of strain inthe 1 direction to strain in the 2 direction for uniaxial stress in the 2 direction

The elastic constants in the principal lamina e directions are defined by:

$$\bar{\sigma}_i = \bar{c}_{ij} \bar{\epsilon}_j \tag{A4}$$

where the overbar denotes quantities referenced to the laminate axes. These constants can be obtained from the lamina constants by coordinate transformations of the stress and strain as follows:

$$\begin{aligned} \sigma_i &= T_{ij} \sigma_j \\ \epsilon_i &= T_{ij} \epsilon_j \end{aligned} \tag{A5}$$

where

$$T_{ij} = [T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Substitution of (A5) and (A4) into (Al) yields:

$$\bar{c}_{ij} = L_{im}\, c_{mn}\, T_{nj} \tag{A6}$$

$$E_T = \frac{1}{\bar{\bar{B}}_{22}}$$

$$G_{LT} = \frac{1}{\bar{\bar{B}}_{33}} \tag{A9}$$

$$\nu_{TL} = -\frac{\bar{\bar{B}}_{12}}{\bar{\bar{B}}_{11}}$$

$$\nu_{LT} = -\frac{\bar{\bar{B}}_{12}}{\bar{\bar{B}}_{22}}$$

The stresses in the kth lamina are now given by

$$\bar{\sigma}_i^{(k)} = \bar{c}_{ij}^{(k)} \bar{\epsilon}_j \tag{A10}$$

and the stress components within any lamina referenced to axes making an angle, α , with the longitudinal and transverse axes are given by:

$$\sigma_i^{(k)} = T_{ij} \bar{\sigma}_j^{(k)} \tag{A11}$$

where the T_{ij} contain α in place of θ .

The elastic constants of equation (A9) and the stresses of equation (A11) have been evaluated for various geometries. The results are discussed in the text.

APPENDIX B

Elastic Stability Filament Wound Cylindrical Shells Under Axial Compression

The stability analysis of filament wound cylindrical shells is a small deflection, classical analysis of anisotropic shells. The work utilizes the results of ref. 6. Under the assumptions that transverse shear strain is negligible and that each bending stiffness is related to the appropriate extensional stiffness in the same fashion that bending and extensional stiffness of homogeneous materials are related, the buckling stress, σ_{cr} , is obtained from equ. A4 of ref. 6 as:

$$\sigma_{cr} = m^2\pi^2\left(\frac{h}{a}\right)^2 \frac{E_L}{12(1-\nu_{LT}\nu_{TL})}\left\{1 + B_1\left(\frac{1}{2\pi^2}\right)\left(\frac{a}{R}\right)^2\left(\frac{n}{m}\right)^2 + \frac{1}{16\pi^4}\left(\frac{E_T}{E_L}\right)\left(\frac{a}{R}\right)^4\left(\frac{n}{m}\right)^4\right\} + \frac{E_L}{\pi^2 m^2\left(\frac{R}{a}\right)^2\left[\frac{E_L}{E_T} + \frac{B_2}{2\pi^2}\left(\frac{a}{R}\right)^2\left(\frac{n}{m}\right)^2 + \frac{1}{16\pi^4}\left(\frac{a}{R}\right)^4\left(\frac{n}{m}\right)^4\right]} \tag{B1}$$

where

$$B_1 = \nu_{LT} + 2(1-\nu_{TL}\nu_{LT})\frac{G_{LT}}{E_L}$$

$$B_2 = \frac{1}{2}\left(\frac{E_L}{G_{LT}} - 2\nu_{TL}\right) \qquad \left(\nu_{LT} = \nu_{TL}\frac{E_T}{E_L}\right)$$

m = longitudinal wave number

n = circumferential wave number

h = shell thickness

a = shell length

R = shell radius

For long cylinders, the longitudinal wave number may be treated as a continuous variable and the buckling stress can be analytically minimized with respect to it. Thus set:

$$\frac{\partial \sigma_{cr}}{\partial m^2} = 0$$

This results in the following equation:

$$m^4 + m^2\left(\frac{E_T}{E_L}\right)\left(\frac{a}{R}\right)^2\left(\frac{1}{\pi^2}\right)\left\{\frac{B_2}{2}n^2 - \frac{R}{h}\left[12\left(\frac{E_L}{E_T}\right)(1-\nu_{LT}\nu_{TL})\right]^{1/2}\right\} + \frac{n^4}{16\pi^4}\left(\frac{a}{R}\right)^4\left(\frac{E_T}{E_L}\right) = 0 \tag{B2}$$

The buckling stress is then determined by selecting an even value of n (excluding n = 2, which is column buckling), finding the associated m value from equ. (B2) and using these values in equ. (B1). The buckling stress is the minimum value of σ_{cr} obtained for the admissible n values.

For the case where n is large, or for an approximate or lower limit solution, it is possible to minimize analytically by treating both wave numbers as continuous variables. This is most easily done by treating m and n/m as the independent variables. Then set

$$\frac{\partial \sigma_{cr}}{\partial m^2} = 0 \tag{B3}$$

$$\frac{\partial \sigma_{cr}}{\partial \left(\frac{n}{m}\right)^2} = 0 \tag{B4}$$

Equations (B1), (B2) and (B4) can be manipulated to yield:

$$\sigma_{cr} = 2E_L\frac{h}{R}\left\{\frac{\frac{1}{12(1-\nu_{LT}\nu_{TL})}\left[1 + B_1\left(\frac{1}{2\pi^2}\right)\left(\frac{a}{R}\right)^2\left(\frac{n}{m}\right)^2 + \frac{1}{16\pi^4}\left(\frac{E_T}{E_L}\right)\left(\frac{a}{R}\right)^4\left(\frac{n}{m}\right)^4\right]}{\frac{E_L}{E_T} + \frac{B_2}{2\pi^2}\left(\frac{a}{R}\right)^2\left(\frac{n}{m}\right)^2 + \frac{1}{16\pi^4}\left(\frac{a}{R}\right)^4\left(\frac{n}{m}\right)^4}\right\}^{1/2} \tag{B5}$$

$$\left(\frac{n}{m}\right)^4 = \frac{\frac{1}{\pi^2}\left(\frac{a}{R}\right)^2 B_1 - B_2}{\frac{1}{16\pi^4}\left(\frac{a}{R}\right)^4\left[B_1 - B_2\frac{E_T}{E_L}\right]} \tag{B6}$$

Equations (B5) and (B6) give an analytical minimum expression for the buckling stress when both wave numbers are treated as continuous variables. Equations (B1) and (B2) were used to numerically minimize with respect to the circumferential wave number to obtain the results described in the body of the paper.

ACKNOWLEDGEMENT

The author is grateful to Mr. N. F. Dow of the Space Sciences Laboratory for his helpful discussions of the subject matter, and to Mr. T. Coffin for his assistance in obtaining the numerical results. This work was supported by the National Aeronautics and Space Administration, under contract No. NASw-817.

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Table 1

Properties of Fibrous Composites

	Material			
Constituent Property	I	II	III	IV
Fiber modulus, E_f, 10^6psi	10.5	10.5	16.0	10.5
Fiber Poissons ratio, V_f	0.20	0.20	0.20	0.20
Binder modulus, E_b, 10^6psi	0.5	0.5	0.5	1.0
Binder Poissons ratio, V_b	0.35	0.35	0.35	0.35
Fiber hollowness ratio, α	1.0	0.8	1.0	1.0
Uniaxial Composite Properties (V_b = 0.3)				
Longitudinal Youngs Modulus, E_1, 10^6psi	7.50	2.80	11.35	7.66
Transverse Youngs Modulus				
upper bound, E_{2U}, 10^6psi	2.66	1.40	2.93	4.20
lower bound, E_{2L}, 10^6psi	1.98	1.28	2.11	3.40
average, E_2, 10^6, psi	2.32	1.34	2.52	3.80
In-Plane Shear Modulus, G_{12}, 10^6psi	0.853	0.519	0.911	1.440
In-Plane Poissons ratio, ν_{21}	0.238	.241	0.238	.239
Transverse Plane Poissons ratio, ν_{23}				
upper bound	0.524	.209	0.549	.465
lower bound	.363	.130	.374	.338
average	.444	.170	.562	.402

FIGURE CAPTIONS (B. W. Rosen)

1. Coordinate System for Laminate Stress Analysis. Symmetric laminate shown.

2. Longitudinal Youngs Modulus of Symmetric Glass Reinforced Plastic Laminates as a Function of the Angle Between the Laminae Principal Axes and the Longitudinal Direction. Results for a triaxial or isotropic laminate are also shown.

3. Shear Modulus of Symmetric Glass Reinforced Plastic Laminates as a Function of the Angle Between the Laminae Principal Axes and the Longitudinal Direction. Results for a triaxial or isotropic laminate and for a longitudinal-transverse laminate are also shown.

4. Longitudinal Youngs Modulus of a Longitudinal-Transverse Laminate as a Function of the Thickness Fraction Oriented with the Fiber Axis in the Longitudinal Direction.

5. Critical Buckling Stress of Helically Wound Glass Reinforced Plastic Cylinders Under Axial Compression.

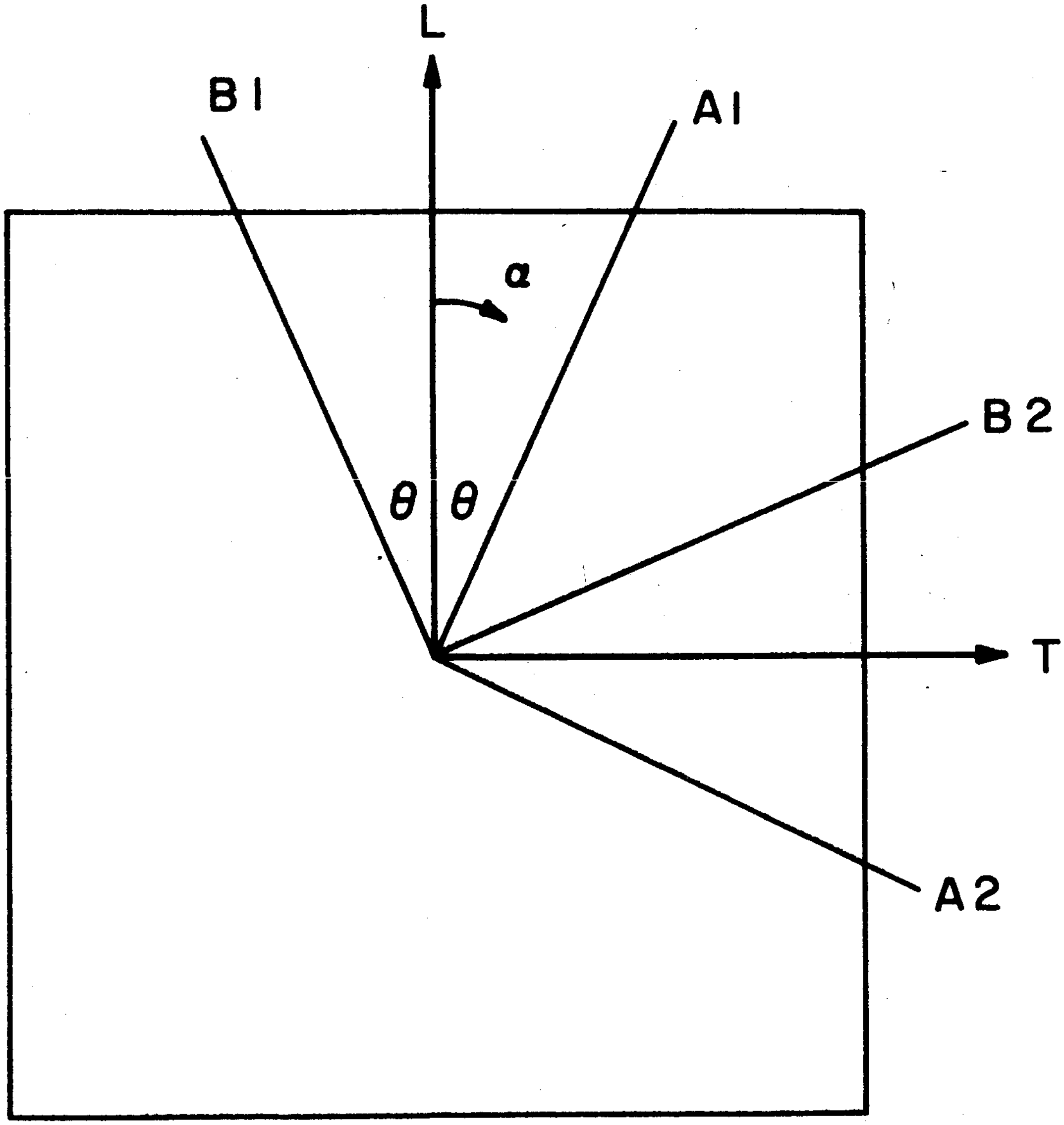
   *Computations for material II are based on equal weight (thus R/t = 102) and resulting stress is multiplied by the ratio of material II thickness to material I thickness so that the comparison is on a load basis.

6. Effect of Variation in Constituent Moduli on Buckling Stress of Cylinder (R/t = 145) in Axial Compression.Material is a Triaxial or Isotropic Laminate and the Reference Cylinder is of Material I.

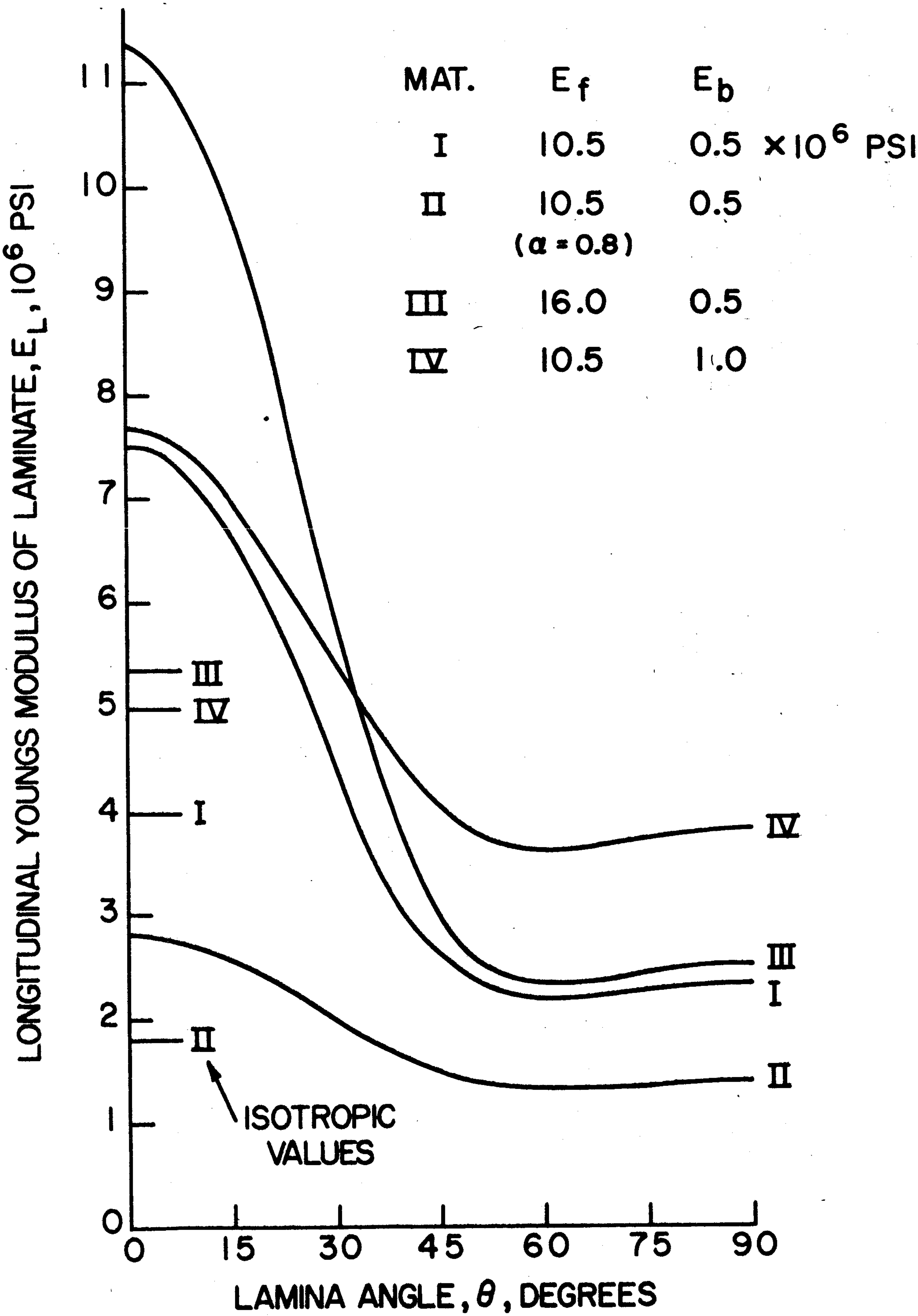
7. Shear Stress on Principal Elastic Axes of Laminae of a Symmetric Laminate Subjected to a Uniaxial Longitudinal Stress, .

8. Influence of Binder Modulus Upon Lamina Shear Stress for a Symmetric 15° Laminate Subjected to a Uniaxial Longitudinal Stress, .

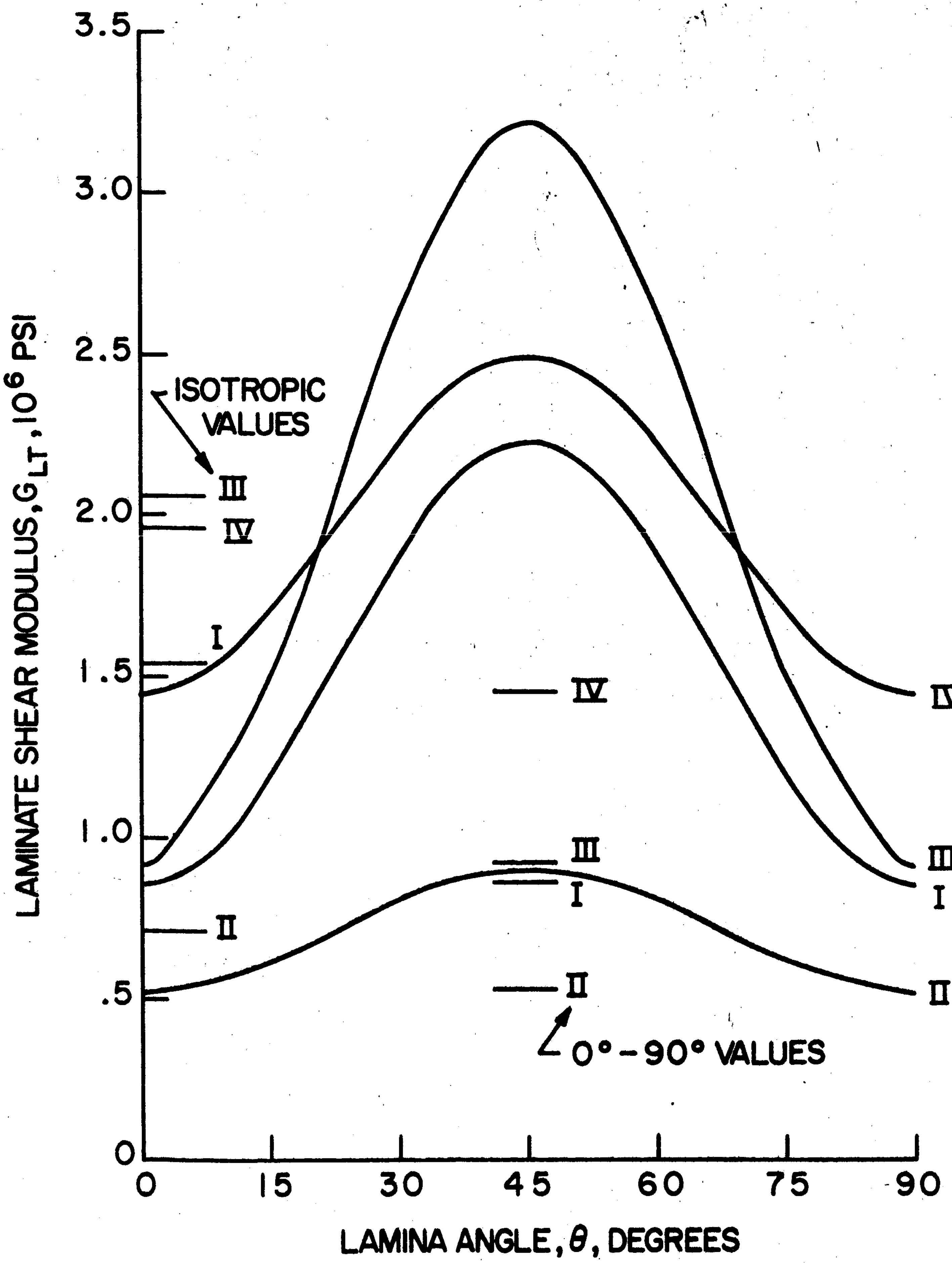
9. Maximum Allowable Stress in Helically Wound Cylindrical Shells (R/t = 145) for Three Different Failure Modes: a) Cylinder Buckling; b) Lamina or In-Plane Shear Failure at 5 ksi Average Stress; c) Transverse Shear Failure at 5 ksi Average Stress.



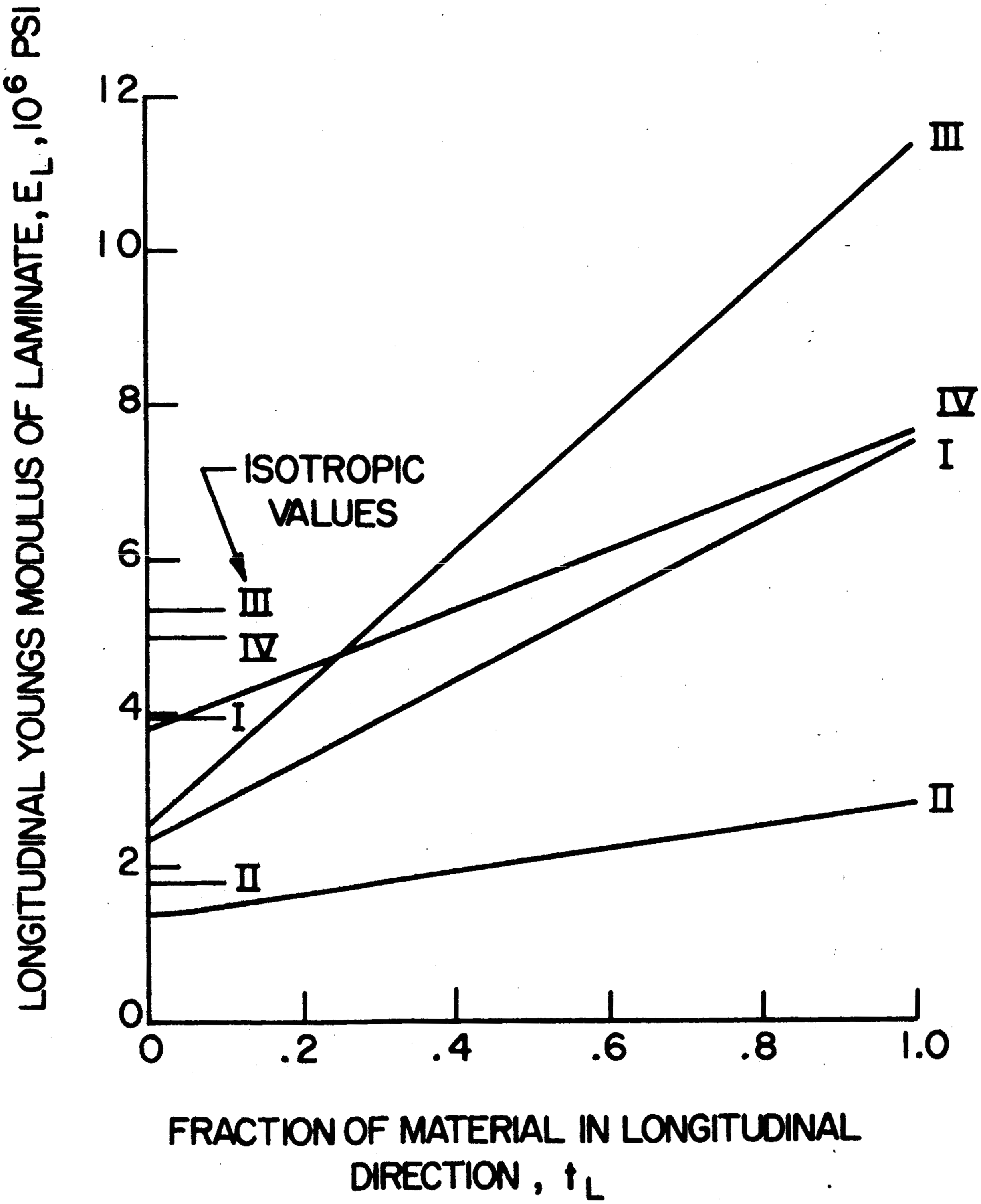
L
B1
A1
α
B2
θ θ
T
A2

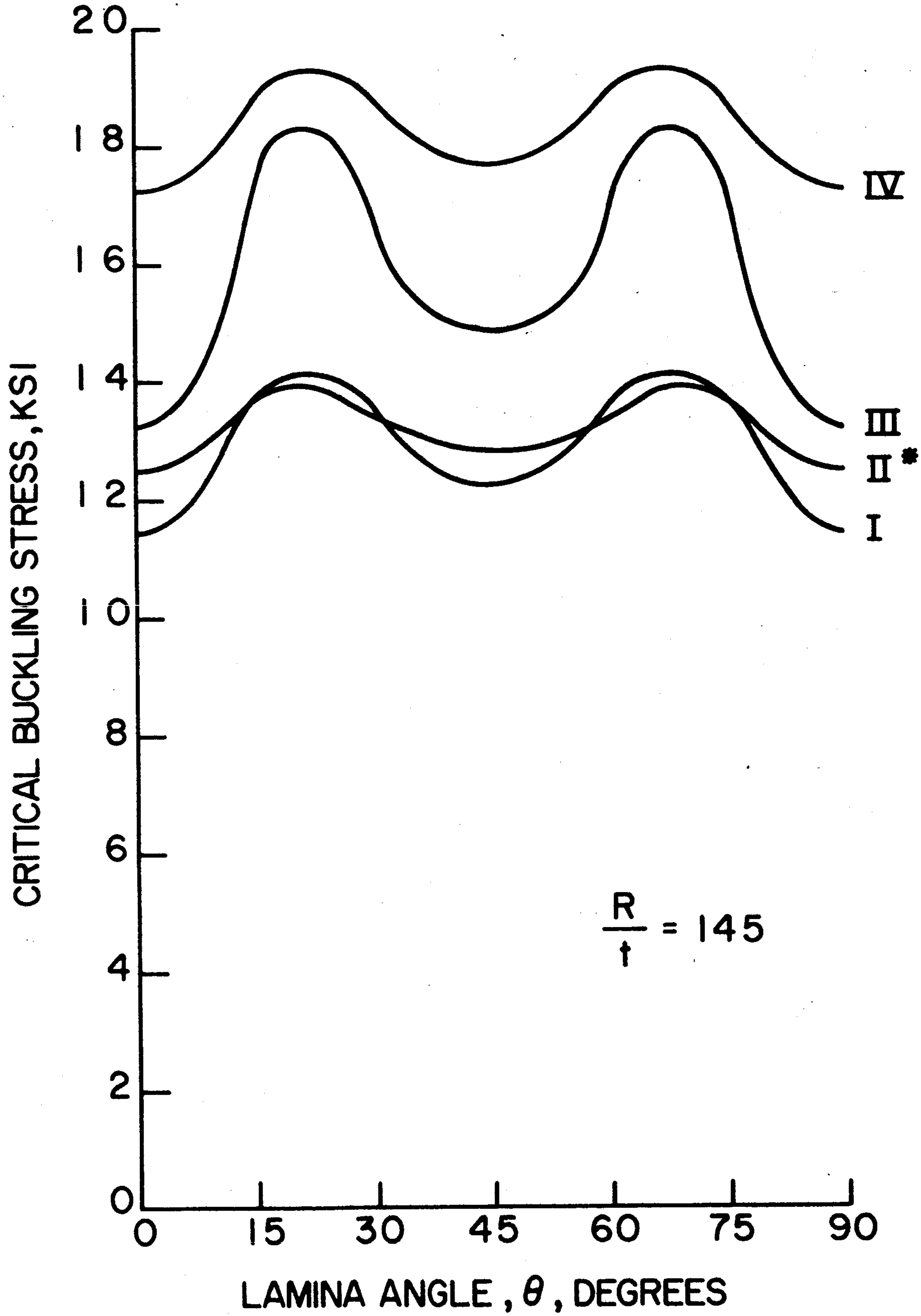


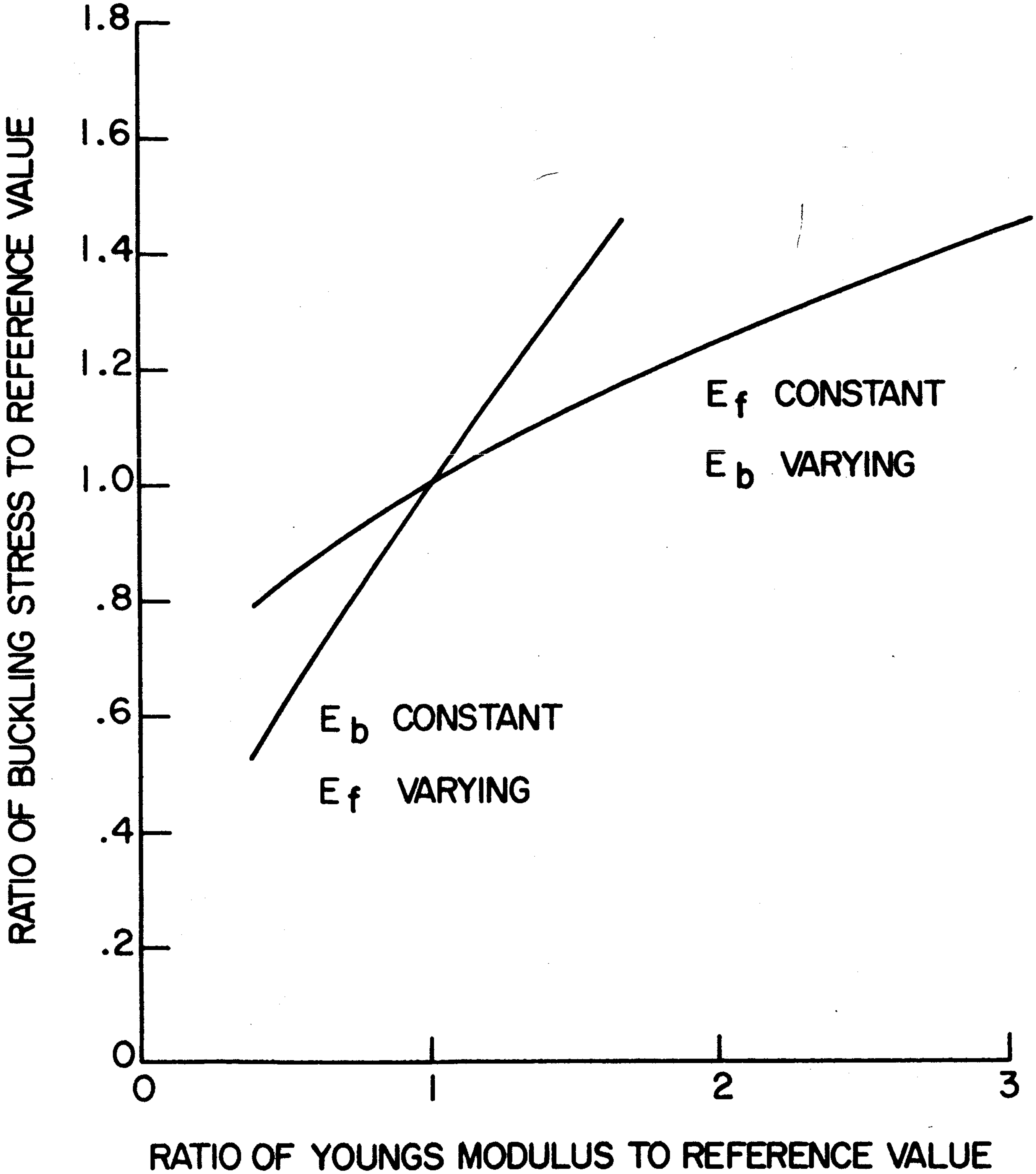
MAT.
E_f
E_b
I
10.5
0.5
$\times 10^6$ PSI
II
10.5
0.5
(α = 0.8)
III
16.0
0.5
IV
10.5
1.0
LONGITUDINAL YOUNGS MODULUS OF LAMINATE, E_L, 10^6 PSI
LAMINA ANGLE, θ, DEGREES
ISOTROPIC VALUES
0
1
2
3
4
5
6
7
8
9
10
11
15
30
45
60
75
90
I
II
III
IV

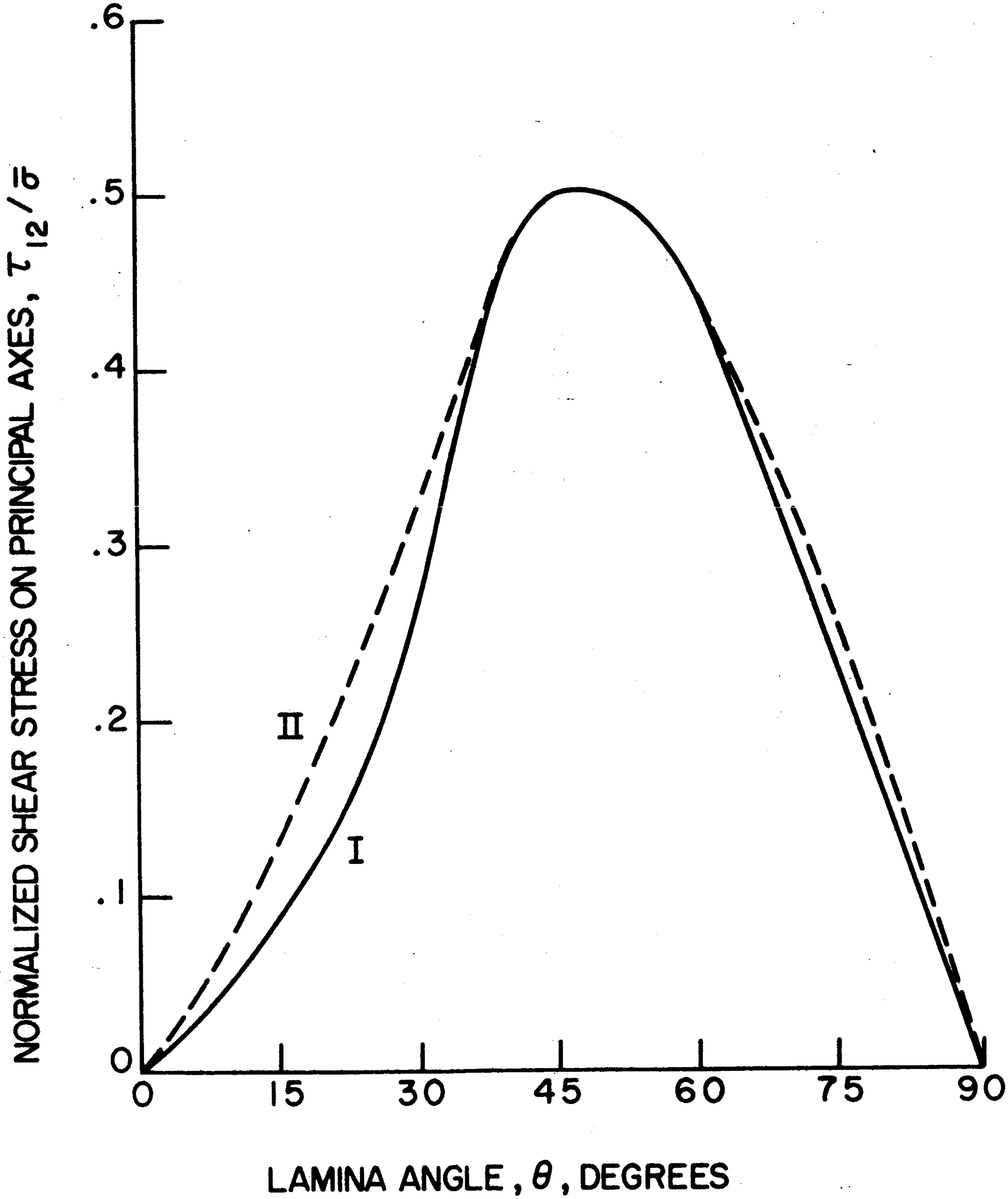


3.5
3.0
2.5
2.0
1.5
1.0
.5
0
LAMINATE SHEAR MODULUS, G_{LT}, 10^6 PSI
ISOTROPIC VALUES
III
IV
I
II
IV
III
I
II
0°-90° VALUES
IV
III
I
II
0
15
30
45
60
75
90
LAMINA ANGLE, θ, DEGREES

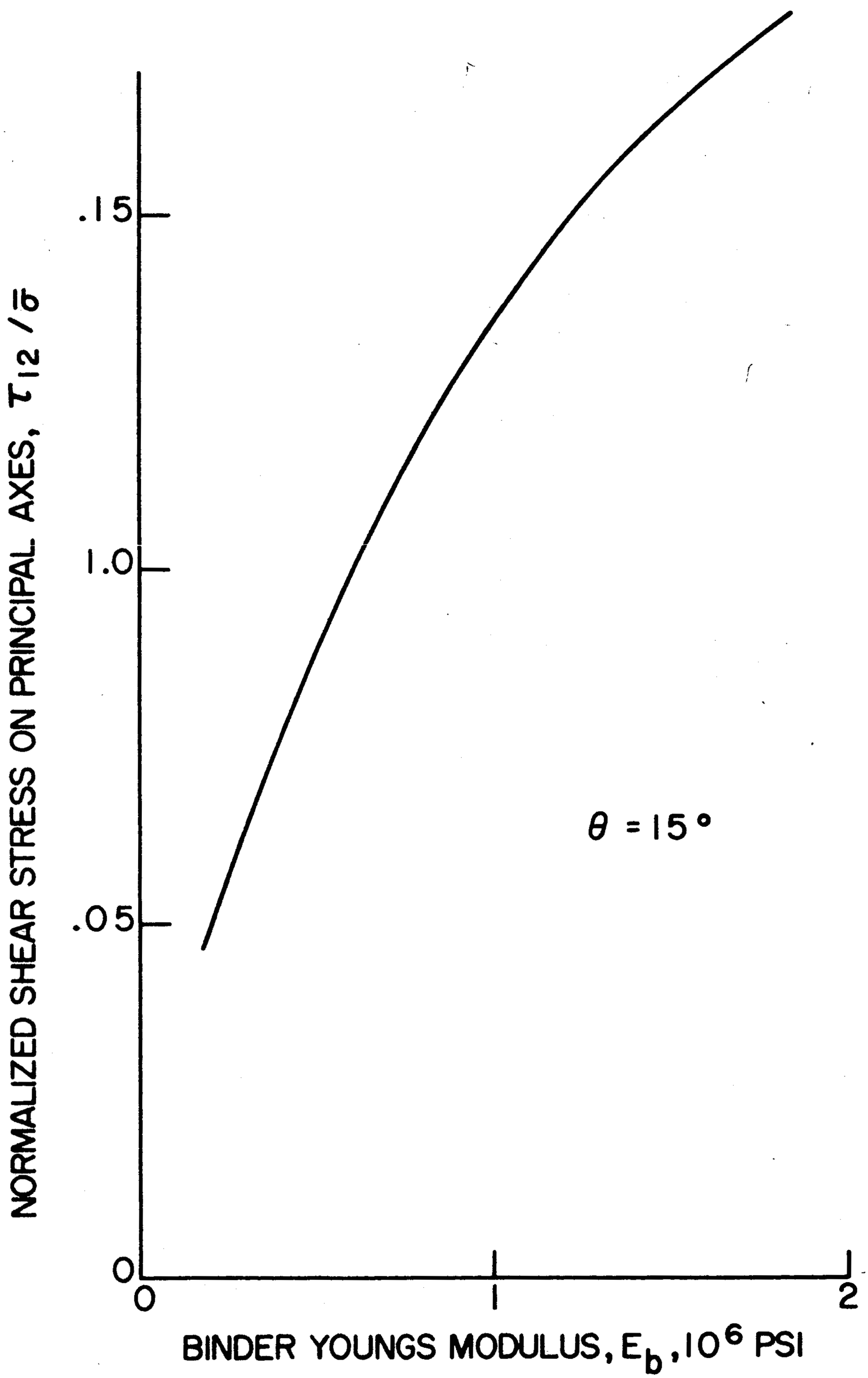

LONGITUDINAL YOUNGS MODULUS OF LAMINATE, E_L, 10^6 PSI
0
2
4
6
8
10
12
ISOTROPIC VALUES
III
IV
I
II
III
IV
I
II
0
.2
.4
.6
.8
1.0
FRACTION OF MATERIAL IN LONGITUDINAL DIRECTION, t_L


CRITICAL BUCKLING STRESS, KSI
LAMINA ANGLE, θ, DEGREES
0
2
4
6
8
10
12
14
16
18
20
0
15
30
45
60
75
90
IV
III
II*
I
R/t = 145

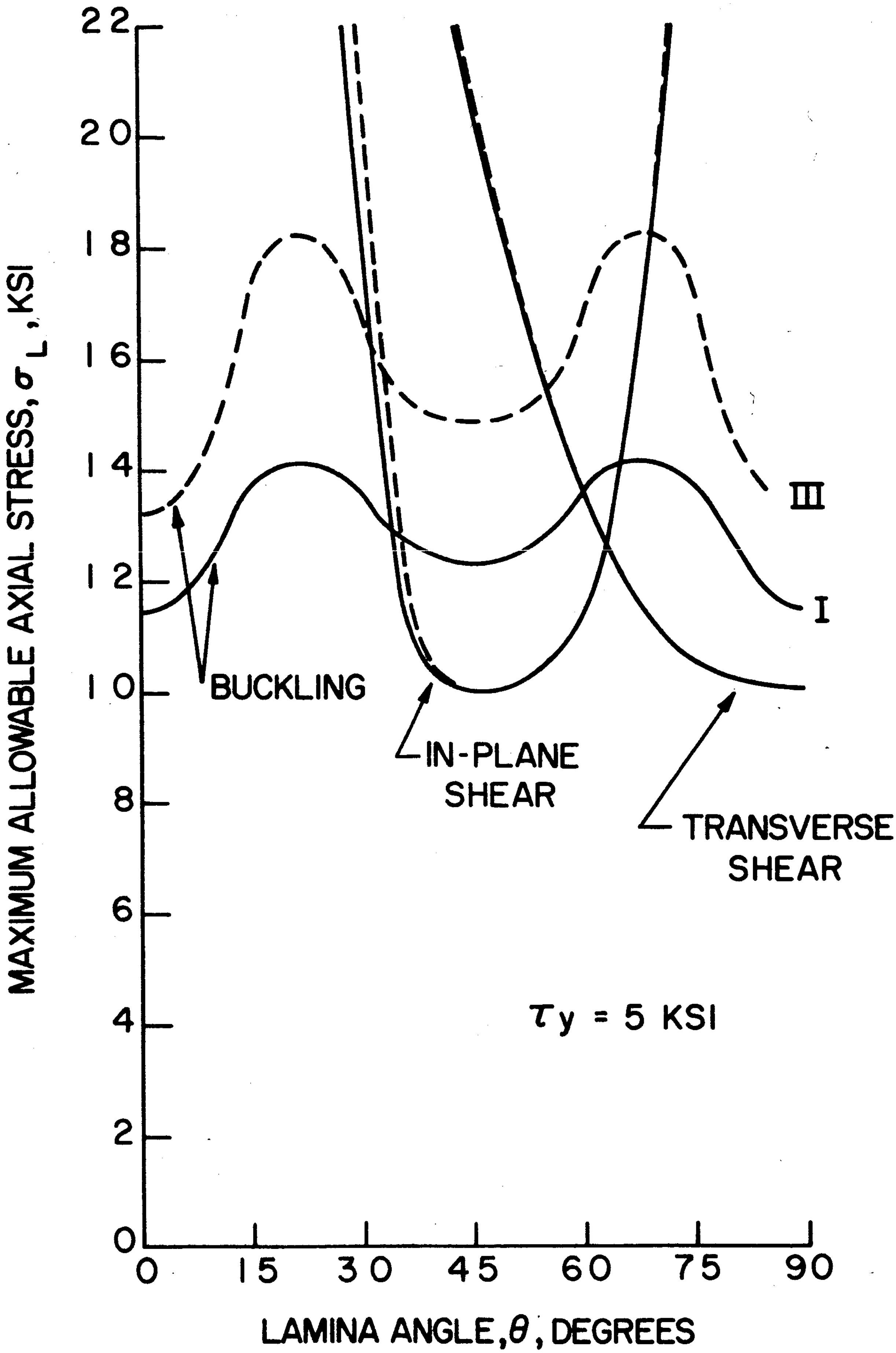

RATIO OF BUCKLING STRESS TO REFERENCE VALUE
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
0
1
2
3
RATIO OF YOUNGS MODULUS TO REFERENCE VALUE
E_f CONSTANT
E_b VARYING
E_b CONSTANT
E_f VARYING



NORMALIZED SHEAR STRESS ON PRINCIPAL AXES, $\tau_{12}/\bar{\sigma}$
.6
.5
.4
.3
.2
.1
0
II
I
0
15
30
45
60
75
90
LAMINA ANGLE, θ, DEGREES



NORMALIZED SHEAR STRESS ON PRINCIPAL AXES, $\tau_{12}/\bar{\sigma}$
.15
1.0
.05
0
θ = 15°
0
1
2
BINDER YOUNGS MODULUS, E_b, 10^6 PSI


22
20
18
16
14
12
10
8
6
4
2
0
0
15
30
45
60
75
90
MAXIMUM ALLOWABLE AXIAL STRESS, σ_L, KSI
LAMINA ANGLE, θ, DEGREES
III
I
BUCKLING
IN-PLANE
SHEAR
TRANSVERSE
SHEAR
τ_y = 5 KSI